In Eq. (9), m refers to the mth natural frequency which is held fixed. Relations similar to those in Eqs. (6a–c) are presented in Ref. 6. This situation occurs because, as in the previous example, the fundamental solution is periodic. The validity of this hypothesis in this problem has been established rigorously for any integer m.

A problem where the previous discussion does not hold true is that of the least weight cantilever beam. The boundary conditions are such that the fundamental solution cannot be made to satisfy boundary conditions for m greater than unity.

Conclusion

It seems reasonable that, if the reference structure in a dynamic optimization problem has eigenfunctions which are periodic, then the least weight structure might also have periodic eigenfunctions. If this is, in fact, the case and if the eigenvalues are also integer multiples of one another, solutions to additional single frequency constraint problems might be generated from "fundamental solutions." If this is true, the value of the objective function which gives an indication of weight savings will not be a function of which natural frequency is held fixed. Although this observation has not been proved rigorously for a general class of problems, it has been illustrated for two simple structures whose natural frequencies are fixed. The subject certainly merits additional investigation.

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Entrainment Equation for Three-Dimensional, Compressible, Turbulent Boundary Layers

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Nomenclature

F = the dimensionless rate of entrainment defined by Eq. (2).

h_1, h_3 = metric coefficients for, respectively, curvilinear coordinates x_1 and x_3.

$\bar{H}$ $= \int_0^\delta \frac{\rho}{\rho_e}\left(1-\frac{u}{u_e}\right) dy \Big/ \delta_{11}$

H_{2k} $= \int_0^\delta \frac{u}{u_e}\, dy \Big/ \int_0^\delta \frac{u}{u_e}\left(1-\frac{u}{u_e}\right) dy$

M = Mach number

s = arc length measured along the x_1 axis according to $ds = h_1\, dx_1$.

u, v, w = boundary-layer velocity components in the x_1, x_2, x_3 directions, respectively.

x_1, x_2, x_3 = orthogonal curvilinear coordinates based on the projections of the outer flow streamlines on the body surface as defined in Fig. 2 of Ref. 5.

y = arc length measured along the x_2 axis according to $dy = dx_2$.

z = arc length measured along the x_3 axis according to $dz = h_3\, dx_3$.

δ = boundary-layer thickness.

δ_1 $= \int_0^\delta \left(1-\frac{\rho u}{\rho_e u_e}\right) dy$

δ_{11} $= \int_0^\delta \left(1-\frac{u}{u_e}\right)\frac{\rho u}{\rho_e u_e}\, dy$

ρ = density

θ_3 $= \int_0^\delta \frac{\rho w}{\rho_e u_e}\, dy$

Subscripts

e = conditions at the outer edge of the boundary layer.

w = conditions at the wall.

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Introduction

THE entrainment theory of Head[1] was originally developed for predicting the flow characteristics of incompressible, turbulent boundary layers. Green[2] extended entrainment theory to compressible, turbulent boundary layers and showed that more accurate results could be obtained by a direct approach as opposed to a transformation to a corresponding incompressible flow. Subsequently, Sumner and Shanebrook[3] modified Green's entrainment theory such that it relies less on empiricism and represents a simpler and more direct extension of Head's incompressible theory to compressible flows. However, Refs. 1–3 are restricted to two-dimensional flow conditions.

Entrainment theory has also been applied to three-dimensional, incompressible, turbulent boundary layers by Cumpsty and Head[4] and by Shanebrook and Hatch[5] who assumed the functions‡ $F(H_{2k})$ and $H_{2k}(\bar{H})$ could be approximated by relations previously developed for two-dimensional flows. The purpose here is to present the entrainment equation for three-dimensional, compressible, turbulent boundary layers and to demonstrate the applicability of relations for $F(H_{2k})$ and $H_{2k}(\bar{H})$, originally developed for two-dimensional incompressible flows, to three-dimensional, compressible, turbulent boundary layers.

Three-Dimensional, Compressible, Entrainment Equation

In an orthogonal curvilinear coordinate system based on the projections of the outer flow streamlines on the body surface, the continuity equation for steady, compressible, turbulent flow may be written in the form[6]

$$[\partial(\rho h_3 u)/\partial x_1] + [\partial(\rho h_1 w)/\partial x_3] + h_1 h_3\, \partial(\rho v)/\partial x_2 = 0 \qquad (1)$$

where the flow variables appear as time-averaged quantities. In the usual fashion Eq. (1) can be integrated across the boundary

‡ The authors have attempted to develop a system of notation that is consistent for both incompressible and compressible, three-dimensional turbulent boundary layers. Thus, the notation adopted here represents a compressible form of the notation used in Ref. 5. The reader should then note that H_{2k} is equivalent to H_2 of Ref. 5 and $\bar{H}$ becomes H_1 of Ref. 5 for incompressible flow conditions.

layer subject to the boundary conditions at the wall

$$x_2 = 0; \quad u = w = 0, \quad v = v_w, \quad \rho = \rho_w$$

and at the boundary-layer edge

$$x_2 = \delta; \quad u = u_e, \quad v = v_e, \quad w = 0, \quad \rho = \rho_e$$

Introducing arc length displacements and the dimensionless rate of entrainment given by

$$F = (1/u_e)(u_e\, \partial\delta/\partial s - v_e) \tag{2}$$

the result of this integration is

$$[\partial(\delta-\delta_1)/\partial s] + \partial\theta_3/\partial z = F + (\rho_w v_w/\rho_e u_e) - (\delta-\delta_1)\times \{[(1-M_e^2)/u_e]\,\partial u_e/\partial s + (1/h_3)\,\partial h_3/\partial s\} - \theta_3 \times \{[(1-M_e^2)/u_e]\,\partial u_e/\partial z + (1/h_1)\,\partial h_1/\partial z\} \tag{3}$$

which is the three-dimensional, compressible, entrainment equation.

Correlations for $F(H_{2k})$ and $H_{2k}(\bar{H})$

Head's[1] two-dimensional, incompressible entrainment theory is based on graphical correlations for $F(H_{2k})$ and $H_{2k}(\bar{H})$ obtained from the experimental data of Newman and of Schubauer and Klebanoff. Standen[7] provided the following analytical approximations to these curves which were subsequently applied to two-dimensional, compressible[3] and three-dimensional, incompressible, turbulent boundary layers[5]

$$F = 0.0306(H_{2k} - 3.0)^{-0.653} \tag{4}$$

$$H_{2k} = 1.535(\bar{H} - 0.7)^{-2.715} + 3.3 \tag{5}$$

Figures 1 and 2 demonstrate the applicability of these relations for three-dimensional, compressible, turbulent boundary layers by comparing with the experimental data of Hall and Dickens[8] and of Rainbird[9] who measured, respectively, the boundary layers on the side wall of a specially constructed supersonic nozzle and on a yawed cone. The rate of entrainment F was determined from

$$F = \frac{1}{\rho_e u_e}\frac{\partial}{\partial s}[\rho_e u_e(\delta-\delta_1)] + \frac{(\delta-\delta_1)}{h_3}\frac{\partial h_3}{\partial s} + \frac{\partial\theta_3}{\partial z} + \theta_3\left[\frac{(1-M_e^2)}{u_e}\frac{\partial u_e}{\partial z} + \frac{1}{h_1}\frac{\partial h_1}{\partial z}\right] \tag{6}$$

which follows from Eq. (3) by combining two terms and setting v_w equal to zero. All terms on the right side of Eq. (6) were determined along streamline B of Ref. 8 from measurements reported by Hall and Dickens[8] except the term, $\partial\theta_3/\partial z$, which was obtained from an integral solution of the governing boundary-layer equations soon to be submitted for publication. It was not possible to perform a similar computation for the flow conditions of Ref. 9 since Rainbird[9] did not present sufficient data for the determination of F from Eq. (6).

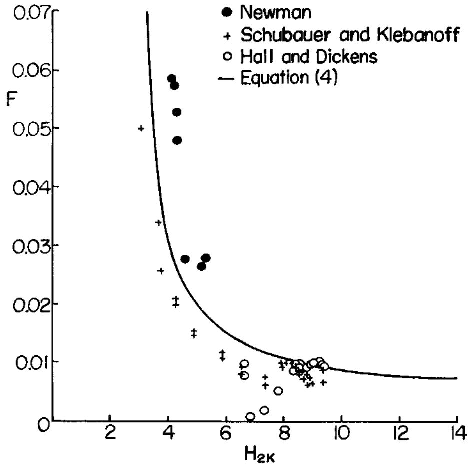

Fig. 1 Comparison of Eq. (4) with three-dimensional, compressible, turbulent boundary-layer data.

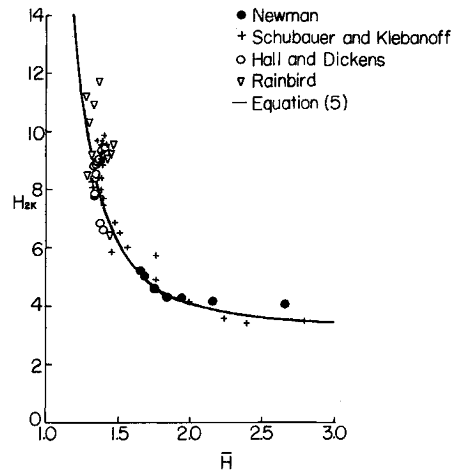

Fig. 2 Comparison of Eq. (5) with three-dimensional, compressible, turbulent boundary-layer data.

Conclusions and Recommendations

All but two of the values of F determined from the data of Hall and Dickens lie within the region originally correlated by Head and therefore it can be concluded that Eq. (4) shows promise for providing an adequate description of the entrainment process in three-dimensional, compressible, turbulent boundary layers. Figure 2 indicates Eq. (5) provides a good correlation for $H_{2k}(\bar{H})$ for the flow conditions investigated in Refs. 8 and 9. Based on these results, it is recommended that Eqs. (3–5) be considered for integral methods whose objective is to predict the growth of three-dimensional, compressible, turbulent boundary layers.

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